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THE CHARACTER MAP IN DEFORMATION QUANTIZATION

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ABSTRACT. The third author recently proved that the Shoikhet–Dolgushev L_∞ -morphism from Hochschild chains of the algebra of smooth functions on manifold to differential forms extends to cyclic chains. Localization at a solution of the Maurer–Cartan equation gives an isomorphism, which we call character map, from the periodic cyclic homology of a formal associative deformation of the algebra of functions to de Rham cohomology. We prove that the character map is compatible with the Gauss–Manin connection, extending a result of Calaque and Rossi on the compatibility with the cap product. As a consequence, the image of the periodic cyclic cycle 1 is independent of the deformation parameter and we compute it to be the A-roof genus of the manifold. Our results also imply the Tamarkin–Tsygan index Theorem.

1. INTRODUCTION

One of the consequences of the formality theorem for cyclic chains [17, 6] is the existence of a *character map* from the periodic cyclic homology of any formal associative deformation A_\hbar of algebras of functions on a manifold M to de Rham cohomology $H^\bullet(M, \mathbb{R}[[\hbar]])$. By Kontsevich’s formality theorem for Hochschild cochains such deformations are classified by Poisson bivector fields in $\hbar\Gamma(M, \wedge^2 TM)[[\hbar]]$. A priori the character map depends on the choice of formality map for Hochschild cochains and cyclic chains. We consider here the Kontsevich L_∞ -morphism of cochains and the Shoikhet–Dolgushev L_∞ -morphism of modules from Hochschild chains to differential forms. The latter was shown by the third author [17] to be compatible with the Rinehart–Connes differential B and the de Rham differential and therefore extends to a morphism of L_∞ -modules from cyclic chains to the de Rham complex. In this paper we study the dependence of the character map on the Poisson bivector field and show that it is compatible with the Gauss–Manin connection. In particular, we show that the image of the cyclic cycle 1 is independent of the Poisson bivector field and can thus be computed at the zero Poisson bracket, where it is given by the A-roof genus of M . This proves one part of Tsygan’s conjecture [15] on the character map (the independence on parameters) and disproves the other (about characteristic classes of foliations) for the Shoikhet–Dolgushev choice of formality map. Our main technical result is that the formality map on periodic chains is compatible with the Gauss–Manin connection. This result is an extension of the Calaque–Rossi theorem [1] on compatibility of the formality map on Hochschild chains with the cap product.

In the rest of the Introduction we recall the main notions (in 1.1) and describe the character map (in 1.2). We then state our main result on the compatibility of

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the character map with the Gauss–Manin connection (Theorem 1.3) and give the formula for the image of the cycle 1 (Corollary 1.5). The proof of the independence of the image of 1 on parameters is easier in the case of regular Poisson structures; we give it in Section 2. The evaluation of the image of 1 at the zero Poisson structure is discussed in Section 3. The case of general Poisson structures is discussed in Section 4, where we also apply our results to obtain a proof of the Tamarkin–Tsygan index Theorem.

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1.1. Cyclic chains as a module over Hochschild cochains and the Gauss–Manin connection. Here we recall some basic notions and fix sign conventions.

Let $C^\bullet(A) = \text{Hom}_k(\bar{A}^{\otimes \bullet}, A)$ be the normalized Hochschild cochain complex of a unital associative algebra A over $k = \mathbb{R}$ or $\mathbb{R}[[\hbar]]$. Here $\bar{A} = A/k1$. The degree-shifted complex $C^{\bullet+1}(A)$ with the Gerstenhaber bracket

$$[\phi, \psi] = \phi \bullet \psi - (-1)^{pq} \psi \bullet \phi, \quad \phi \in C^{p+1}(A), \quad \psi \in C^{q+1}(A),$$

$$\phi \bullet \psi(a_1, \dots, a_{p+q+1}) = \sum_{i=1}^{p+1} (-1)^{(i-1)q} \phi(a_1, \dots, a_{i-1}, \psi(a_i, \dots, a_{i+q}), \dots, a_{p+q+1}),$$

is a differential graded Lie algebra; the differential is the bracket $b = [\mu, \bullet]$ with the product $\mu \in C^2(A)$. The Lie bracket induces a Lie bracket on the cohomology $HH^{\bullet+1}(A)$ of $C^{\bullet+1}(A)$.

The complex of normalized Hochschild chains $C_\bullet(A) = A \otimes \bar{A}^{\otimes (-\bullet)}$ is a graded module over the dgla $C^\bullet(A)$. It is concentrated in non-positive degrees. The action of a cochain $\phi \in C^{p+1}(A)$ on a chain $a = (a_0, \dots, a_n) \in C_{-n}(A)$ is

$$\begin{aligned} L_\phi a &= (-1)^p \sum_{i=0}^{n-p} (-1)^{ip} (a_0, \dots, a_{i-1}, \phi(a_i, \dots, a_{i+p}), \dots, a_n) \\ &\quad + (-1)^p \sum_{i=n-p+1}^n (-1)^{in} (\phi(a_i, \dots, a_n, a_0, \dots, a_{p-n+i-1}), \dots, a_{i-1}). \end{aligned}$$

The Hochschild differential on $C_\bullet(A)$ is the action of the product $b = L_\mu$. The module property implies that $b \circ L_\phi - (-1)^p L_\phi \circ b = L_{b\phi}$ for all $\phi \in C^{p+1}(A)$, so that the action induces an action of the graded Lie algebra $HH^{\bullet+1}(A)$ on the homology $HH_\bullet(A)$ of (C_\bullet, b) . There is a second differential B of degree -1 on this complex (anti-)commuting with b : the Rinehart–Connes differential

$$B(a_0, \dots, a_n) = \sum_{i=0}^n (-1)^{in} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1}).$$

The induced action of $\phi \in HH^p(A)$ on $HH_\bullet(A)$ obeys the homotopy formula $L_\phi = B \circ I_\phi - (-1)^p I_\phi \circ B$, where $I_\phi: C_{-n}(A) \rightarrow C_{-n+p}(A)$ is the internal multiplication, defined on chains by

$$(1) \quad I_\phi(a_0, \dots, a_n) = (a_0 \phi(a_1, \dots, a_p), a_{p+1}, \dots, a_n).$$

Here is a version of the homotopy formula on chains due to Getzler [8], who extended Rinehart’s formula [12] for $p = 1$.

Lemma 1.1. *Let $\phi \in C^p(A)$ and let $H_\phi: C_{-n}(A) \rightarrow C_{-n+p-2}(A)$ be the map*

$$H_\phi(a_0, \dots, a_n) = \sum_{i=0}^{n-p} \sum_{j=0}^{n-p-i} (-1)^{i(n-p+1)+j(p+1)} (1, a_{n+1-i}, \dots, a_n, a_0, \dots, a_j, \phi(a_{j+1}, \dots, a_{j+p}), \dots, a_{n-i})$$

Then H_ϕ commutes with B and

$$L_\phi = [B, I_\phi] - H_\phi B + [B, H_\phi].$$

Here $[,]$ denotes the graded commutator in $\text{End}_k(C_\bullet(A))$.

Proof. Both $H_\phi \circ B$ and $B \circ H_\phi$ vanish since $(1, 1, \dots) = 0$ in the normalized complex. The formula for L_ϕ can be proved by explicit calculations, see [12, 8]. \square

The periodic cyclic chain complex is the complex of Laurent polynomials $PC_\bullet(A) = C_\bullet(A)[u, u^{-1}]$ in an indeterminate u of degree 2 and u -linear differential $D = b + uB$. Lemma 1.1 and the identities $[b, I_\phi] = I_{b\phi}$, $[B, H_\phi] = 0$ imply that the u -linear extension of L_ϕ obeys

$$(2) \quad L_\phi = \frac{1}{u}([D, \hat{I}_\phi] - \hat{I}_{b\phi}), \quad \hat{I}_\phi = I_\phi + uH_\phi,$$

on the periodic cyclic complex.

Let (A_t) be a one-parameter family of algebras given by an algebra over $k[t]$ equal to $A[t]$ as a $k[t]$ -module. Then the product has the form $\mu + \gamma(t)$, where μ is the product in $A = A_0$ and $\gamma(t)$ is a solution of the Maurer–Cartan equation $b\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)] = 0$. As noticed by Getzler [8], eq. (2) implies that there exists a connection, called Gauss–Manin connection, on the periodic cyclic cohomology viewed as a vector bundle over the parameter space. The parallel translation is given at the level of chains by the following formula.

Lemma 1.2. (Getzler) *Suppose that $c(t) \in PC_\bullet(A_t)$ is a solution of the differential equation*

$$\frac{d}{dt}c(t) + \frac{1}{u}\hat{I}_{\dot{\gamma}(t)}c(t) = 0,$$

and that $c(0)$ is a cyclic cycle. Then $c(t)$ is a cyclic cycle for all t and its class in $PH_\bullet(A_t)$ depends only on the class of $c(0)$.

The statement follows from (2) and the fact that $\dot{\gamma}(t) = \frac{d}{dt}\gamma(t)$ is a 2-cocycle in $C^2(A_t)$.

Getzler also showed that this formula defines a flat connection on the periodic cyclic cohomology of any family of algebras. Tsygan [16] extended the Gauss–Manin connection to a flat superconnection at the level of chains, see also [7].

1.2. The character map. Let now $k = \mathbb{R}$ and $A = C^\infty(M)$ be the algebra of smooth functions on a manifold M and $C^\bullet(A)$ be the complex of multidifferential Hochschild cochains. All operations described above are defined on this subcomplex of the Hochschild complex.

Kontsevich [9], in his proof of the formality theorem, constructed an L_∞ -quasi-isomorphism \mathcal{U} from the dgla $T^{\bullet+1}(M) = \Gamma(M, \wedge^{\bullet+1}TM) \simeq HH^{\bullet+1}(A)$, with

Schouten–Nijenhuis bracket¹ and zero differential, to $C^{\bullet+1}(A)$. The first component \mathcal{U}_1 is the Hochschild–Kostant–Rosenberg (HKR) quasi-isomorphism. Shoikhet [13] proved Tsygan’s formality conjecture for chains on $M = \mathbb{R}^n$ by giving a morphism \mathcal{V} of L_∞ -modules over $T^{\bullet+1}(M)$ from $C_\bullet(A)$ to the dg module of differential forms $\Omega^{-\bullet}(M, \mathbb{R})$ with zero differential. The action of $\gamma \in T^p(M)$ on $\Omega^{-\bullet}$ is given by the Lie derivative $L_\gamma = d \circ \iota_\gamma - (-1)^p \iota_\gamma \circ d$, where d is the de Rham differential and the internal multiplication of vector fields is extended to multivector fields by the rule $\iota_\gamma \iota_\eta = \iota_{\gamma \wedge \eta}$. The L_∞ -action of $T^{\bullet+1}(M)$ on Hochschild chains is the pull-back of the action of cochains on chains by Kontsevich’s morphism \mathcal{U} . Shoikhet’s result was extended to general manifolds by Dolgushev [5]. The first component \mathcal{V}_0 of the L_∞ -morphism \mathcal{V} is the HKR map $(a_0, \dots, a_n) \mapsto (1/n!)a_0 da_1 \cdots da_n$ and induces an isomorphism in homology if we interpret $C_{-n}(A)$ as the complex of ∞ -jets around the small diagonal of M^{n+1} . Finally, it was shown by the third author [17] that the Shoikhet–Dolgushev morphism commutes with the Connes differential B on chains and the de Rham differential on differential forms and thus defines a morphism of L_∞ -modules over $T^{\bullet+1}(M)$ from the periodic cyclic complex $PC_\bullet(A)$ to the de Rham complex $(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])[u, u^{-1}], ud)$.

For any solution of the Maurer–Cartan equation $[\pi, \pi] = 0$ in $\hbar T^2(M)[[\hbar]]$, i.e., a formal Poisson bivector field, we get a solution $\mathcal{U}_0^\pi = \sum_{n=1}^\infty \frac{1}{n!} \mathcal{U}_n(\pi, \dots, \pi)$ of the Maurer–Cartan equation $b\mathcal{U}_0^\pi + \frac{1}{2}[\mathcal{U}_0^\pi, \mathcal{U}_0^\pi] = 0$ in $C^2(A)[[\hbar]]$. The Maurer–Cartan equation is the associativity of the star product $f \star g = fg + \mathcal{U}_0^\pi(f \otimes g)$ on $A[[\hbar]]$. Let A_\hbar be the algebra $(A[[\hbar]], \star)$. The twists at π of \mathcal{U}, \mathcal{V} are L_∞ -morphisms $\mathcal{U}^\pi, \mathcal{V}^\pi$ defined by

$$\begin{aligned} \mathcal{U}_n^\pi(\gamma_1, \dots, \gamma_n) &= \sum_{m=0}^\infty \frac{\hbar^m}{m!} \mathcal{U}_{n+m}(\gamma_1, \dots, \gamma_n, \pi, \dots, \pi), \quad n \geq 1, \\ \mathcal{V}_n^\pi(\gamma_1, \dots, \gamma_n) &= \sum_{m=0}^\infty \frac{\hbar^m}{m!} \mathcal{V}_{n+m}(\gamma_1, \dots, \gamma_n, \pi, \dots, \pi), \quad n \geq 0. \end{aligned}$$

The twist \mathcal{U}^π is an \hbar -linear L_∞ -quasiisomorphism from $T^{\bullet+1}(M)[[\hbar]]$ with Poisson differential $[\pi, \]$ to $C^{\bullet+1}(M)[[\hbar]]$ with Hochschild differential $b_\star = b + L_{\mathcal{U}_0^\pi}$ calculated with the star product. Similarly \mathcal{V}^π is an L_∞ -morphism of modules over the dgla $(T^{\bullet+1}(M), [\pi, \])$ from $(PC_\bullet(A)[[\hbar]], b_\star + uB)$ to $(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])[u^{\pm 1}], L_\pi + ud)$. We refer to [9, 15, 13, 5, 17] for more details.

Since $L_\pi = d \circ \iota_\pi - \iota_\pi \circ d$, the map $\alpha \mapsto e^{\iota_\pi/u} \alpha$ is an isomorphism of complexes

$$(\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])[u, u^{-1}], L_\pi + ud) \rightarrow (\Omega^{-\bullet}(M, \mathbb{R}[[\hbar]])[u, u^{-1}], ud)$$

Passing to cohomology, we obtain the *character map*

$$PH_\bullet(A_\hbar) \rightarrow H^{-\bullet}(M, \mathbb{R}[[\hbar]])[u, u^{-1}],$$

of modules over the Poisson cohomology, induced by

$$\tilde{\mathcal{V}}_0^\pi = e^{\iota_\pi/u} \circ \mathcal{V}_0^\pi,$$

from the periodic cyclic cohomology of the deformed algebra A_\hbar to de Rham cohomology.

¹The Schouten–Nijenhuis bracket is the Lie bracket on vector fields and is extended to general multivector fields by the rule $[\alpha \wedge \beta, \gamma] = \alpha \wedge [\beta, \gamma] + (-1)^{pq} \beta \wedge [\alpha, \gamma]$, $\alpha \in T^p(M), \beta \in T^q(M)$. With this sign convention the HKR homomorphism sends $\xi_1 \wedge \cdots \wedge \xi_p$ to the Hochschild cocycle $a_1 \otimes \cdots \otimes a_p \mapsto (1/p!) \sum_{\sigma \in S_p} \text{sgn}(\sigma) \xi_{\sigma(p)}(a_1) \cdots \xi_{\sigma(1)}(a_p)$

1.3. Dependence on parameters. Suppose we have a one-parameter family of solutions $\pi(t)$ of the Maurer–Cartan equation, with, say, polynomial dependence on t , for example $\pi(t) = t\pi$ for some Poisson structure π . We then obtain a one-parameter family of star-products whose reduction modulo \hbar^N depends polynomially on t for any N . The Gauss–Manin connection allows us to identify the periodic cyclic cohomology for different values of the parameter, so we can study the dependence on t of the character map.

Theorem 1.3. *Let A_t be the quantum algebra of functions on M associated with the Poisson bracket $\pi(t)$. Suppose $c(t)$ is a cycle in $PC_{-n}(A_t)$, horizontal with respect to the Gauss–Manin connection, i.e., such that*

$$\frac{d}{dt}c(t) + \frac{1}{u}\hat{I}_{\mathcal{U}_1^{\pi(t)}(\dot{\pi}(t))}c(t) = 0, \quad \hat{I}_\gamma = I_\gamma + uH_\gamma.$$

Then the class of $\tilde{\mathcal{V}}_0^{\pi(t)}(c(t))$ in $\oplus_m H^{n+2m}(M, \mathbb{R}[[\hbar]])u^m$ is independent of t .

The proof of this theorem is based on an identity extending the “compatibility with cap product” result of Calaque and Rossi [1]. To formulate the result we introduce the modified internal multiplication of a multivector field γ on differential forms: $\hat{\iota}_\gamma = \iota_\gamma + u\frac{1}{2}dL_\gamma$. It is clear that the Cartan formula $L_\gamma = [d, \hat{\iota}_\gamma]$ still holds.

Proposition 1.4. *Let π be a Poisson bivector field on M , $A = C^\infty(M)$ and $\gamma \in \Gamma(M, \wedge^p TM)$. There exist maps*

$$\mathcal{V}_I^\pi(\gamma): C_\bullet(A) \rightarrow \Omega^{-\bullet+p-1}(M, \mathbb{R}[[\hbar]]), \quad \mathcal{V}_H^\pi(\gamma): C_\bullet(A) \rightarrow \Omega^{-\bullet+p-3}(M, \mathbb{R}[[\hbar]]),$$

linear in γ , such that if $[\pi, \gamma] = 0$,

$$\hat{\iota}_\gamma \circ \mathcal{V}_0^\pi - \mathcal{V}_0^\pi \circ \hat{I}_{\mathcal{U}_1^\pi(\gamma)} + u\mathcal{V}_1^\pi(\gamma) = (L_\pi + ud) \circ X^\pi(\gamma) + X^\pi(\gamma) \circ (b_\star + uB),$$

with homotopy $X^\pi(\gamma) = \mathcal{V}_I^\pi(\gamma) + u\mathcal{V}_H^\pi(\gamma)$.

Proposition 1.4 is our main technical result and is proven in Section 4. The operators \mathcal{V}_I and \mathcal{V}_H are given explicitly in terms of integrals over configuration spaces.

Given this proposition, the proof of the theorem is a simple calculation. With the notation $\pi = \pi(t)$, $\dot{\pi} = \frac{d}{dt}\pi(t)$, $b_\star(t) = b + L_{\pi(t)}$ we have:

$$\begin{aligned} \frac{d}{dt}\mathcal{V}_0^\pi + \frac{1}{u}\hat{\iota}_{\dot{\pi}} \circ \mathcal{V}_0^\pi &= \mathcal{V}_1^\pi(\dot{\pi}) + \frac{1}{u}\hat{\iota}_{\dot{\pi}} \circ \mathcal{V}_0^\pi \\ &= \frac{1}{u}(\mathcal{V}_0^\pi \circ \hat{I}_{\mathcal{U}_1^\pi(\dot{\pi})} + (L_\pi + ud) \circ X^\pi(\dot{\pi}) \\ &\quad + X^\pi(\dot{\pi}) \circ (b_\star(t) + uB)) \end{aligned}$$

If $c(t)$ is a cycle obeying the differential equation we get

$$\left(\frac{d}{dt} + \frac{1}{u}\hat{\iota}_{\dot{\pi}}\right)(\mathcal{V}_0^\pi c(t)) = \left(d + \frac{1}{u}L_\pi\right)(X^\pi(\dot{\pi})c(t))$$

This formula can be rewritten as

$$\frac{d}{dt}\left(e^{\hat{\iota}_{\dot{\pi}}/u}\mathcal{V}_0^\pi c(t)\right) = d\left(e^{\hat{\iota}_{\dot{\pi}}/u}X^\pi(\dot{\pi})c(t)\right).$$

Thus the de Rham class of $e^{\hat{\iota}_{\dot{\pi}}/u}\mathcal{V}_0^\pi c(t)$ is independent of t . Moreover, since $e^{\hat{\iota}_{\dot{\pi}}/u} = (1 - \frac{1}{2u}d\hat{\iota}_{\dot{\pi}}d)e^{\hat{\iota}_{\dot{\pi}}/u}$, and $e^{\hat{\iota}_{\dot{\pi}}/u}\mathcal{V}_0^\pi c(t)$ is a cocycle, we may replace $\hat{\iota}_{\dot{\pi}}$ by $\iota_{\dot{\pi}}$ in the exponential.

Remark. The identity of Proposition 1.4 reduces at $u = 0$ to the identity

$$\iota_\gamma \circ \mathcal{V}_0^\pi - \mathcal{V}_0^\pi \circ I_{\mathcal{U}_1^\pi(\gamma)} = L_\pi \circ \mathcal{V}_I^\pi(\gamma) + \mathcal{V}_I^\pi(\gamma) \circ b_\star.$$

This is the statement of compatibility with the cap product ($I_\phi = \phi \cap$) proved in [1].

Remark. Geometrically one should think of the periodic cyclic complex $PC_\bullet(A_\hbar)$ as the fibre of a trivial vector bundle on the Maurer–Cartan variety of formal deformations of the product. This vector bundle carries the Gauss–Manin connection $d + \theta$, where θ is a one-form with values in the endomorphisms of C_\bullet . The value of this one-form on a tangent vector $\gamma \in \text{Ker}(b_\star)$ is

$$\theta_\gamma = \frac{1}{u} \hat{I}_\gamma = \frac{1}{u} I_\gamma + H_\gamma.$$

The statement of Proposition 1.4 is that the Shoikhet–Dolgushev quasi-isomorphism intertwines, up to homotopy, this connection and the connection $d + \frac{1}{u} \hat{I}$ on the trivial vector bundle on the Maurer–Cartan variety of Poisson structure with fibre $\Omega^\bullet(M, \mathbb{R}[[\hbar]])[u, u^{-1}]$.

1.4. The image of the cycle 1. For any unital algebra A , 1 is a cyclic cycle and thus also a periodic cyclic cycle.

Corollary 1.5. *Let $\pi \in \hbar\Gamma(M, \wedge^2 TM)[[\hbar]]$ be a formal Poisson bivector field. The image of the cyclic cycle $1 \in (C^\infty(M)[[\hbar]], \star)$ by the character map is $\hat{A}_0(M) + u\hat{A}_2(M) + u^2\hat{A}_4(M) + \dots$, where $\hat{A}_i(M) \in H^i(M)$ are the components of the A -roof genus of M .*

Indeed, $\pi(t) = t\pi$ is a Poisson bivector field for any t . By Theorem 1.3 we can evaluate the class of $\tilde{\mathcal{V}}_0^{t\pi}(1)$ at $t = 0$. We do this in Section 3.

2. SPECIAL CASE: REGULAR POISSON STRUCTURES

In this section, we will illustrate the above results for regular Poisson structures. In this special case, there is a simple proof of \hbar -independence that does not require the more intricate Theorem 4.1. The results of this section are not needed anywhere else in the paper.

Proposition 2.1. *For \mathcal{V} the Shoikhet–Dolgushev morphism and π a regular Poisson structure, the de Rham cohomology class of*

$$\tilde{\mathcal{V}}_0(1) = e^{\hbar\iota_\pi/u} \mathcal{V}_0^\pi(1)$$

is independent of \hbar .

In general, let us introduce the handy notation $\tilde{\mathcal{V}}_j = e^{\hbar\iota_\pi/u} \mathcal{V}_j^\pi$. It is then easy to see that the first L_∞ relations reduce to the following

$$\begin{aligned} u d\tilde{\mathcal{V}}_0(\alpha) &= \tilde{\mathcal{V}}_0((b_\star + uB)\alpha) \\ u d\tilde{\mathcal{V}}_1(\gamma; \alpha) + (-1)^{|\gamma|} (L_\gamma + \hbar\iota_{[\pi, \gamma]}; \alpha) \tilde{\mathcal{V}}_0(\alpha) &= (-1)^{|\gamma|} \tilde{\mathcal{V}}_1(\gamma; (b_\star + uB)\alpha) \\ &\quad + \hbar\tilde{\mathcal{V}}_1([\pi, \gamma]; \alpha) + (-1)^{|\gamma|} \tilde{\mathcal{V}}_0(L_{\mathcal{U}(\gamma)}\alpha) \end{aligned}$$

For the proof of the proposition, first note that the subcomplex $(T_\parallel^\bullet, [\pi, \cdot]) \subset (T^\bullet, [\pi, \cdot])$ of multivector fields tangential to the symplectic foliation is locally acyclic. Concretely, in a small enough neighbourhood around any point, the manifold looks like a product of a symplectic space and one with trivial Poisson structure.

It follows that there is a good covering $\{U_i\}_{i \in I}$ of M , a collection of (locally defined) tangential vector fields $\xi = \{\xi_i\}_{i \in I}$, and a collection of (locally defined) functions $f = \{f_{ij}\}_{i \neq j \in I}$, $f_{ij} = -f_{ji}$, such that

$$\begin{aligned} \pi &= [\pi, \xi_i] && \text{on } U_i \\ \xi_i - \xi_j &= [\pi, f_{ij}] && \text{on } U_i \cap U_j. \end{aligned}$$

In other words, the class $[\pi]$ in Poisson cohomology can be represented by the cocycle δf in the Poisson-Čech double complex, where δ is the Čech-differential.

Next compute on U_i :

$$\begin{aligned} \frac{d}{d\hbar} \tilde{\mathcal{V}}_0(1) &= \frac{d}{d\hbar} e^{\hbar \iota_\pi / u} \sum_{j \geq 0} \frac{\hbar^j}{j!} \mathcal{V}_j(\pi, \dots, \pi; 1) \\ &= \iota_\pi \tilde{\mathcal{V}}_0(1)/u + \tilde{\mathcal{V}}_1(\pi; 1) \\ &= \iota_\pi \tilde{\mathcal{V}}_0(1)/u + \tilde{\mathcal{V}}_1([\pi, \xi_i]; 1) \\ &= \iota_\pi \tilde{\mathcal{V}}_0(1)/u + \left(-\frac{1}{\hbar} L_{\xi_i} - \iota_\pi / u\right) \tilde{\mathcal{V}}_0(1) + \frac{u}{\hbar} d\tilde{\mathcal{V}}_1(\xi_i; 1) \\ &= +\frac{1}{\hbar} d \left(-\iota_{\xi_i} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1(\xi_i; 1) \right) \end{aligned}$$

The class of this cocycle in Čech-de Rham cohomology is the same as that of the Čech-degree 1 chain (denoting $\xi_{ij} = \xi_i - \xi_j$)

$$\begin{aligned} \frac{1}{\hbar} \left(-\iota_{\xi_{ij}} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1(\xi_{ij}; 1) \right) &= \frac{1}{\hbar} \left(-\iota_{\xi_{ij}} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1([\pi, f_{ij}]; 1) \right) \\ &= \frac{u}{\hbar^2} \left(d\iota_{f_{ij}} \tilde{\mathcal{V}}_0(1) + u d\tilde{\mathcal{V}}_1(f_{ij}; 1) - \tilde{\mathcal{V}}_0(L_{\mathcal{U}_1(f_{ij})} 1) \right) \\ &= \frac{u}{\hbar^2} \left(d\iota_{f_{ij}} \tilde{\mathcal{V}}_0(1) + u d\tilde{\mathcal{V}}_1(f_{ij}; 1) \right. \\ &\quad \left. - \frac{1}{u} \tilde{\mathcal{V}}_0((uB + b_\star) \mathcal{U}_1(f_{ij})) \right) \\ &= \frac{u}{\hbar^2} d \left(\iota_{f_{ij}} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1(f_{ij}; 1) - \tilde{\mathcal{V}}_0(\mathcal{U}_1(f_{ij})) \right) \end{aligned}$$

This cochain in turn describes the same Čech-de Rham cohomology class as the Čech-degree 2 cochain (denoting $f_{ijk} = f_{ij} + f_{jk} + f_{ki}$)

$$\frac{u}{\hbar^2} \left(f_{ijk} \tilde{\mathcal{V}}_0(1) + u \tilde{\mathcal{V}}_1(f_{ijk}; 1) - \tilde{\mathcal{V}}_0(\mathcal{U}_1(f_{ijk})) \right)$$

We claim that this expression is zero. Concretely, it follows from the following lemma.

Lemma 2.2. *For the Shoikhet–Dolgushev morphism \mathcal{V} and a regular Poisson structure π the following hold*

- For any functions f, a_0, \dots, a_n

$$\mathcal{V}_1^\pi(f; a_0, \dots, a_n) = 0.$$

- For any Poisson-central function f and any functions a_1, \dots, a_n

$$\mathcal{V}_0^\pi(f, a_1, \dots, a_n) = f \mathcal{V}_0^\pi(1, a_1, \dots, a_n).$$

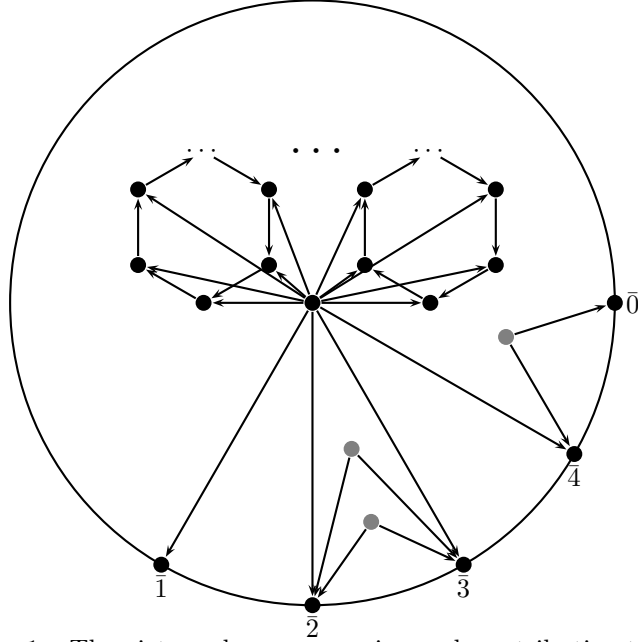


FIGURE 1. The picture shown a generic graph contributing to \mathcal{V}_0^π in the regular case. The grey vertices correspond to the (fiber-wise constant) Poisson structure $\hbar\pi$. Thus putting $\hbar = 0$ the grey vertices disappear and the only contributions come from the wheels.

Proof. For the proof, we assume familiarity with the construction of Dolgushev and Shoikhet, see [5, 13]. In the globalization, we choose the Grothendieck connection such that the flat lift of π is constant. Then the only graphs that can contribute are products of wheels, cf. Figure 1. By just counting the number of edges, the first of the statements in the Lemma follows. The second statement follows, because the vertex of f can at most be hit by edges from a π -vertex, but those graphs yield zero contribution since f is leafwise constant ($[\pi, f] = 0$). \square

Noting that $[\pi, f_{ijk}] = 0$, and hence also $\mathcal{U}_1(f_{ijk}) = f_{ijk}$, Proposition 2.1 is proven. \square

3. THE SHOIKHET–DOLGUSHEV MAP FOR $\pi = 0$

This section is devoted to the following result, needed in the proof of Corollary 1.5.

Proposition 3.1. *For \mathcal{V} the Shoikhet–Dolgushev morphism, $\pi = 0$, and c a cyclic cycle in $C_\bullet(A)(u)$*

$$[\tilde{\mathcal{V}}_0^{\pi=0}(1)] = \hat{A}_u(M) Co(c).$$

Here

$$\hat{A}_u(M) := \hat{A}_0(M) + u\hat{A}_2(M) + u^2\hat{A}_4(M) + \dots$$

with $\hat{A}_i(M) \in H^i(M)$ being the components of the A -roof genus of M . Furthermore Co is the Connes map identifying periodic cyclic and de Rham cohomology,

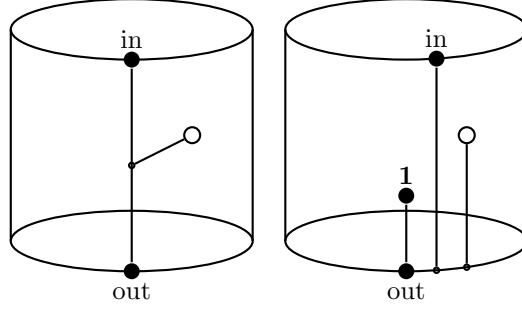


FIGURE 2. The operations I (cap product, left) and H (right) in the Kontsevich Soibelman operad.

concretely on chains

$$Co(a_0, \dots, a_n) = a_0 da_1 \wedge \dots \wedge da_n.$$

Proof. The only contributing graphs are unions of wheels and edges directly connecting the center vertex with vertices on the boundary, as shown in Figure 1 (without the grey vertices). It is not hard to see that the latter (direct) edges produce the factor $Co(c)$. Let us turn to the wheels. Consider a single wheel with j vertices. It has a number (weight) w_n and a differential form α_n associated to it. The contribution of all combinations of wheels is hence

$$[\tilde{\mathcal{V}}_0(1)] = [e^{\sum_{j \geq 2} w_j \alpha_j}].$$

From the details of the globalization procedure (and tedious sign conventions) it can be seen that

$$\alpha_j = (-1)^{j(j-1)/2} u^j \text{tr}(R^j)$$

where R is the curvature 2-form of some affine torsion free connection on M . Furthermore, the weights w_j can be computed explicitly. A simple reflection argument shows that $w_j = 0$ if j is odd. For j even it was shown by M. van den Bergh [19] (and the authors in [18]) that $w_j = -(-1)^{j(j-1)/2} \frac{B_j}{2j!}$.² Hence it follows that

$$[\tilde{\mathcal{V}}_0(1)] = [e^{-\sum_{j \geq 1} \frac{B_{2j}}{4j(2j)!} u^j \text{tr}(R^{2j})}] = \det^{\frac{1}{2}} \left(\frac{uR/2}{\sinh(uR/2)} \right) = \hat{A}_u(M).$$

□

4. COMPATIBILITY OF THE CYCLIC SHOIKHET–DOLGUSHEV MAP WITH CAP PRODUCTS

4.1. KS -Operations on the Hochschild complex and multivector fields.

Let KS be the Kontsevich-Soibelman operad, see [10, 11] for details. It acts naturally on Hochschild chains and cochains $(C^\bullet(A), C_\bullet(A))$, but not on multivector fields and differential forms. However, using graphs and Kontsevich integrals, one can define such an action explicitly, up to higher homotopies. The maps \mathcal{U}, \mathcal{V} respect these actions up to higher homotopies given again by explicit integral expressions. For this paper only the two operations I and H shown in Figure 2 are relevant.

²This differs slightly from the formula in [19] since we included the symmetry factor j of the wheel in the weight.

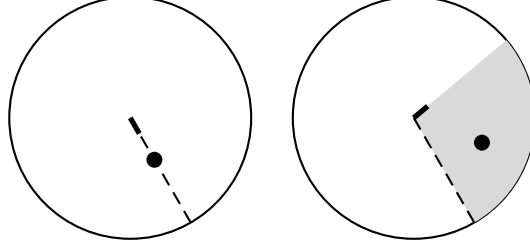


FIGURE 3. The subspaces U_I (left) and U_H (right). On the left, the inner and outer framings are aligned and the vertex is restricted to this common line. On the right, the framings are not aligned and the vertex may move in the shaded region between them. Note that the framings are not fixed, but part of the configuration.

We will actually restrict to these operations and first homotopies here, the full construction, i.e., all other KS -operations and higher homotopies, will be described elsewhere. The operation I corresponds to the cap product. On the Hochschild side, the corresponding operation is given by eqn. (1). On the multivector fields the operation corresponds to the insertion

$$I_\gamma \alpha = \iota_\gamma \alpha.$$

The operation I generates the action of the Hochschild cochains (resp. multivector fields) on the Hochschild chains (resp. differential forms) up to a homotopy H , given by the right graph in Figure 2 above. On the Hochschild side, this operation is described explicitly by the formula in Lemma 1.1. On the differential forms side, with $I_\gamma = \iota_\gamma$ and $b = 0$, one has the well known Cartan formula

$$[d, \iota_\gamma] - L_\gamma = [0, H_\gamma] = 0.$$

Here the homotopy H_γ does not play a role since the differential is zero. However, for reasons apparent later, we should take it to be the following operation

$$H_\gamma \alpha = \frac{1}{2} dL_\gamma \alpha.$$

4.2. Compatibility of \mathcal{V} with the KS -action. One can use graphs and Kontsevich integrals to construct a homotopy KS map between the Hochschild cochains/chains and multivector fields/differential forms. We will first restrict to the local case (the manifold is \mathbb{R}^n or $\mathbb{R}_{\text{formal}}^n$). To each generator of the homotopy KS operad, one associates a subspace of an appropriate configuration space, similarly to [10]. The respective component (“Taylor coefficient”) of the homotopy KS -map is then the sum of graphs with coefficient the integral of the weight form over the respective subspace. The details will be described elsewhere. We will only need here the components of the map corresponding to the operations I and H from above, and only in the π -twisted case. The proof will be done “by hand”. The relevant configuration space is a compactification of the space of configurations of points in the punctured framed disk, modulo rotations. Here the puncture is at the center 0 of the disk and “framed” means that (i) there is a distinguished point on the boundary of the disk and (ii) there is a distinguished direction at the center. The framing is part of the configuration. The subspaces U_I and U_H corresponding

to the operation I and H are shown in Figure 3. To be concrete, let us define the Taylor components, directly in the twisted case.

$$\begin{aligned} \mathcal{V}_{Op}^\pi(\gamma; a_0, \dots, a_n) &= \\ &= \sum_{m \geq 0} \frac{\hbar^m}{m!} \sum_{\Gamma \in G(m+1, n)} \left(\int_{\pi_1^{-1}(U_{Op}) \subset C_\Gamma} \omega_\Gamma \right) D_\Gamma(\gamma, \pi, \dots, \pi; a_0, \dots, a_n). \end{aligned}$$

Here Op is one of the operations I, H . The π_1 is the projection to the configuration space of the first vertex (and the framings) alone, forgetting the positions of all the other vertices. The remainder of the notation is defined in appendix A. The operation \mathcal{V}_I has already been found by D. Calaque and C. Rossi [1], cf. the Remark before section 1.4. The main result of this section is the following theorem.

Theorem 4.1. *The operation $\mathcal{V}_H^\pi(\gamma)$ satisfies*

$$\begin{aligned} (3) \quad \mathcal{V}_0^\pi H_{\mathcal{U}_1^\pi(\gamma)} - H_\gamma \mathcal{V}_0^\pi(-1)^{\deg_H} - \mathcal{V}_1^\pi(\gamma) + (-1)^{\deg_f} d\mathcal{V}_I^\pi(\gamma) + \mathcal{V}_I^\pi(\gamma) B(-1)^{\deg_H} = \\ = -(-1)^{|\gamma|} \mathcal{V}_H^\pi(\gamma) b_\star - L_\pi \mathcal{V}_H^\pi(\gamma) - \mathcal{V}_H^\pi([\pi, \gamma]) \end{aligned}$$

where \deg_H and \deg_f are the Hochschild and differential form grading operators.

Proof. This is a standard Kontsevich-Stokes proof. Concretely, one applies Stokes' Theorem to the integrals

$$\int_{\pi_1^{-1}(U_H)} d\omega_\Gamma = \int_{\partial\pi_1^{-1}(U_H)} \omega_\Gamma.$$

The left hand side vanishes since the weight forms are closed. Computing the right hand side yields some polynomial relations between weights, which are equivalent to the equation in the Theorem. In our case, the following codimension 1 boundary strata appear.

- (1) $k \geq 2$ interior vertices can approach each other in the interior. By a Kontsevich lemma, the contribution is zero unless $k = 2$. If the vertex 1 (corresponding to γ) is one of the two vertices collapsing, the result is the term $\mathcal{V}_H^\pi([\pi, \gamma])$. If not, the result is zero by the Jacobi identity $[\pi, \pi] = 0$.
- (2) Some interior vertices (except 1) approach the center 0. This produces the term $L_\pi \mathcal{V}_H^\pi(\gamma)$, see [13].
- (3) A cluster of k interior vertices, one of them is 1, approaches the center 0. If $k > 1$ this yields zero contribution by (a slight variation of) Kontsevich's Lemma. If $k = 1$, i.e., only the vertex 1 approaches 0, we claim that the contribution is $\pm H_\gamma \mathcal{V}_0^\pi$. One might take this as the definition of H_γ . But let us nevertheless show that the term produced equals $\pm \frac{1}{2} dL_\gamma$. Let α be the angle between the inner and outer framing. Start by integrating out the position of vertex 1. As in the previous stratum, this produces a term L_γ , but there remains a factor α in the weight form due to the restricted integration domain. Integrating out the inner framing yields an overall de Rham derivative (as in [17]), weighted by a factor $\frac{1}{2} = \int_0^1 \alpha d\alpha$.
- (4) The vertex 1 (corresponding to γ) can approach the line connecting 0 and $\bar{0}$. In this case one can easily integrate out the framing at the center vertex (see [17]) and obtains the term $d\mathcal{V}_I^\pi(\gamma)$.
- (5) The vertex 1 can approach the line corresponding to the framing at 0. The result is equal to $\mathcal{V}_I^\pi(\gamma) B$.

- (6) The inner and outer framing align. This leaves no restriction on the position of the vertex 1 and hence produces $\mathcal{V}_1^\pi(\gamma)$.
- (7) Vertices on the boundary circle and zero or more interior vertices approach each other. If the vertex 1 (corresponding to γ) is not in the approaching cluster, this produces the term $\mathcal{V}_H^\pi(\gamma)b_\star$.
- (8) If the vertex 1 is in the approaching cluster, the resulting term is equal to $\mathcal{V}_0^\pi H_{\mathcal{U}_1^\pi(\gamma)}$.

We apologize for not being very precise about signs. \square

To globalize the above result, one has to replace multivector fields, multidifferential operators, differential forms and Hochschild chains by their Fedosov resolutions and apply the above map fiberwise, as usual. See, e.g., [5] for a thorough account of the standard globalization procedure.

4.3. (Remainder of) Proof of Proposition 1.4. The above result show that the equality in Proposition 1.4 is satisfied in linear order in u . For the sign, note that in our case $\gamma = \hat{\pi}$ is even, as are all forms and Hochschild chains involved.

The quadratic order in u of the equality in Proposition 1.4 asserts that the homotopy \mathcal{V}_H^π is again compatible with de Rham and Rinehart differential, i.e. that

$$d\mathcal{V}_H^\pi(\gamma) + \mathcal{V}_H^\pi(\gamma)B = 0.$$

Actually, both terms vanish separately.³ We restrict ourselves to the local case: Integrating out the inner framing in the explicit integral formula for $\mathcal{V}_H^\pi(\gamma)$ shows that $\mathcal{V}_H^\pi(\gamma) = d(\dots)$ and hence the first term vanishes. On the other hand, consider $\mathcal{V}_H^\pi(\gamma)B$. It consists of terms having 1 inserted at the marked point on the boundary of the disk (the outer framing). Hence there is no angle form depending on the outer framing and the integral vanishes. \square

4.4. Relation to the Tamarkin-Tsygan index Theorem. As observed by D. Tamarkin and B. Tsygan [14] the results on cyclic formality may be used to prove an algebraic index theorem as follows. Let π be a Poisson structure and $\pi_t = t\pi$ for $t \in \mathbb{R}$. Let A_t be the space $A = C^\infty(M)[[\hbar]]$ with the star product corresponding to π . Let $c \in PH_\bullet(A_1)$ be a cyclic homology class. By parallel transport wrt. the Gauss-Manin connection (see Theorem 1.3) one obtains a family of cyclic homology classes $c(t) \in PH_\bullet(A_t)$ such that $c(1) = c$. In particular $c(0) \in PH_\bullet(A)$. By Connes isomorphism (see section 3) $PH_\bullet(A) \cong H^\bullet(M)(u)$. We denote the image of $c(0)$ by $ch(c) \in H^\bullet(M)(u)$.

On the other hand one has a map from periodic cyclic to cyclic homology $PH_\bullet(A_1) \rightarrow CH_\bullet(A_1)$. On chains, this map is given by sending positive powers of u to zero. Furthermore, the zeroth cyclic homology is isomorphic to the zeroth Hochschild homology $CH_0(A_1) \cong HH_0(A_1)$. By the Shoikhet-Dolgushev morphism \mathcal{V}^π the latter is, in turn, isomorphic to zeroth Poisson homology $HP_0(M)$. Assume now that there is a volume form Ω on M and π is a unimodular with $\text{div}_\Omega \pi = 0$.⁴ Then there is a natural map $HP_0(M) \rightarrow \mathbb{C}$ by $f \mapsto \int_M f\Omega$.

³This could be derived from the *KS* relations, together with the facts $B \circ H = H \circ B = 0$ and the vanishing of the homotopy corresponding to the composition. However, we do it by hand here.

⁴A Poisson structure is called unimodular if its divergence with respect to some (and hence every) volume form is a Hamiltonian vector field. If this is the case, rescaling the volume form by the exponential of the Hamiltonian function yields a volume form with respect to which the

Following Tamarkin and Tsygan, let us denote the composition by⁵

$$I: PH_0(A_1) \rightarrow CH_0(A_1) \xrightarrow{\mathcal{V}^\pi} HP_0(M) \rightarrow \mathbb{C}$$

$$I(c) = \int_M \mathcal{V}_0^\pi(c) \Omega.$$

By Theorem 1.3 we know that

$$[\mathcal{V}_0^\pi(c)] = [e^{-\iota_\pi/u} \mathcal{V}_0^{\pi=0}(c(0))].$$

Furthermore, by Proposition 3.1

$$\mathcal{V}_0^{\pi=0}(c(0)) = \hat{A}_u(M) ch(c).$$

Hence we arrive at the following Theorem, known as the Tamarkin-Tsygan index Theorem. The statement can be found in [14].

Theorem 4.2. *Let M be a compact manifold, π a Poisson structure, Ω a volume form on M with $\text{div}_\Omega \pi = 0$ and $c \in PH_0(A_\hbar)$. Then*

$$I(c) = \int_M \hat{A}_u(M) ch(c) e^{\iota_\pi/u} \Omega.$$

APPENDIX A. GRAPHS AND WEIGHTS

We briefly recall here some standard fact about Kontsevich graphs and their weights. For more details, see [9].

Definition A.1. The set $G(m, n)$, $m, n \in \mathbb{N}_0$ of *Shoikhet graphs* consists of directed graphs Γ such that

- (1) The vertex set of Γ is

$$V(\Gamma) = \{0, \dots, m\} \cup \{\bar{0}, \dots, \bar{n}\}$$

where the vertices $\{1, \dots, m\}$ will be called the *type I* vertices and the vertices $\{\bar{0}, \dots, \bar{n}\}$ the *type II* vertices.

- (2) Every edge $e = (v, w) \in E(\Gamma)$ starts at a type I vertex, i.e., $v \in \{1, \dots, m\}$.
- (3) No edge ends at vertex 0.
- (4) For each type I vertex j , there is an ordering given on

$$Star(j) = \{(j, w) \mid (j, w) \in E(\Gamma), w \in V(\Gamma)\}.$$

- (5) There are no double edges, i.e., edges (j, w) occurring twice in $E(\Gamma)$.
- (6) There are no tadpoles, i.e., edges of type (j, j) .

Let us next define the (Shoikhet) weight w_Γ of $\Gamma \in G(m, n)$. It is an integral of a certain differential form over a compact manifold with corners, the *configuration space* C_Γ .

$$(4) \quad w_\Gamma = \int_{C_\Gamma} \omega_\Gamma$$

Poisson structure is divergence-free. If $\{ , \}$ denotes the Poisson bracket, then the last condition is equivalent to $\int_M \{f, g\} \Omega = 0, \forall f, g$.

⁵Note that by form degree reasons, terms with positive powers of u , though present in the integrand, do not contribute to I .

Definition A.2. The enlarged Shoikhet configuration space \tilde{C}_Γ is the Fulton-MacPherson-Axelrod-Singer compactification of the space of embeddings

$$(z_0, \dots, z_m, z_{\bar{0}}, \dots, z_{\bar{n}}) : V(\Gamma) \rightarrow D$$

of the vertex set $V(\Gamma)$ of Γ into the closed unit disk $D = \{z \in \mathbb{C}; |z| \leq 1\}$, together with a distinguished direction at vertex 1, such that

- (1) All type I vertices are mapped to the interior of D , i.e. $z_j \in D^\circ$ for $j = 0, \dots, m$.
- (2) All type II vertices are mapped to the boundary of D , i.e. $z_{\bar{j}} \in \partial D$ for $j = 0, \dots, n$.
- (3) The type II vertices occur in counterclockwise increasing order on the circle, i.e., $0 < \arg \frac{z_{\bar{1}}}{z_0} < \dots < \arg \frac{z_{\bar{n}}}{z_0} < 2\pi$.

The *Shoikhet configuration space* C_Γ is the quotient of \tilde{C}_Γ under the action of the automorphism group of the unit disk $PSU(1, 1)$.

The differential form ω_Γ that is integrated over configuration space can be expressed as a product of one-forms, one for each edge in Γ .

$$\omega_\Gamma = \bigwedge_{(0,v) \in E(\Gamma)} \alpha_1(v) \bigwedge_{j=1}^n \bigwedge_{(j,v) \in E(\Gamma)} \alpha(j, v)$$

Here the one-form $\alpha_0(v)$ is the differential of the (hyperbolic) angle between the framing at the vertex z_0 and the hyperbolic geodesic from z_0 to v . Similarly, the angle $\alpha(j, v)$ is the differential of the hyperbolic angle between the hyperbolic straight lines (z_j, v) and (z_j, z_0) .

To any graph $\Gamma \in G(m+1, n)$ as above one can associate a function D_Γ taking m vector fields $\gamma_1, \dots, \gamma_m$ and $n+1$ functions and returning a d -differential form where d is the degree of vertex 0 in Γ . It is defined such that for a constant d -vector field γ_0

$$\begin{aligned} (5) \quad & (-1)^d \iota_{\gamma_0} D_\Gamma(\gamma_1 \otimes \dots \otimes \gamma_m; a_1, \dots, a_n) = \\ & = \sum_{\varphi: E(\Gamma) \rightarrow [d]} \prod_{j=0}^m (\partial_{\varphi(f_1^j)} \partial_{\varphi(f_2^j)} \dots \gamma_j^{\varphi(e_1^j) \varphi(e_2^j) \dots}) \prod_{k=1}^n (\partial_{\varphi(f_1^{\bar{k}})} \partial_{\varphi(f_2^{\bar{k}})} \dots a_k). \end{aligned}$$

Here the sum runs over all maps φ from the edge set of Γ to the set $\{1, \dots, d\}$. The edges incoming to a vertex v are denoted by f_1^v, f_2^v, \dots and the edges outgoing by e_1^v, e_2^v, \dots .

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